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Thermodynamics of anisotropic phonon systems in superfluid helium

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Abstract
The thermodynamic functions of anisotropic phonon systems in superfluid helium are calculated for all levels of anisotropy. The results show that the thermodynamic functions of strongly anisotropic phonon systems are essentially different from isotropic ones. It is shown that for strongly anisotropic phonon systems, in thermodynamic equilibrium, the energy density of high-energy phonons, $\varepsilon/k_{\mathrm{B}} \geqslant 10$ K, is more than ten times higher than in a cone with the same total energy density and with the Bose–Einstein distribution for an isotropic system. The stability curve for anisotropic phonon systems is derived and it is shown that strongly anisotropic phonon systems are thermodynamically stable over a wide temperature range.

1. Introduction

At low enough temperatures, He II can be considered as a gas of phonons moving in the superfluid component of the liquid helium. When there is complete equilibrium, the phonon gas is isotropic in the frame of the superfluid component. For such a phonon gas there is no special direction as all directions are equivalent. When a weak heat flux or propagation of sound is present, the phonon system becomes weakly anisotropic. Such isotropic and weakly anisotropic phonon systems in superfluid ^{4}He have been studied for many years. However, in superfluid helium it is possible to create quasiparticle systems which are strongly anisotropic [1], where most of phonons move in one direction. The thermodynamical properties of such strongly anisotropic phonon systems have not been considered until this present paper. We will see that strongly anisotropic phonon systems are qualitatively and quantitatively different to isotropic ones. For example two groups of phonons emerge instead of one, and the energy density is much larger, at the same temperature, in anisotropic phonon systems.

A strongly anisotropic phonon system can be created by a short current pulse in a heater which is immersed in superfluid helium. If the helium is at a sufficiently low temperature ($T \leqslant 50$ mK), then the thermal excitations of helium can be neglected, and the phonon pulse moves in a phonon vacuum. The phonon pulse has a net momentum along the direction normal to the heater; this defines the anisotropy axis, and in momentum space, the distribution of occupied states is strongly anisotropic. The transverse dimensions of the phonon pulse are about those of the heater, when it is near the heater (i.e. typically 1 mm $\times$ 1 mm). The longitudinal dimension is ct_{p}, where c is the velocity of sound and t_{p} is the duration of the heater pulse.

If the relaxation time of this system is much less than the shortest time for the properties of the system to change significantly, then it can be in a quasi-thermodynamic equilibrium. This can be described by an equilibrium distribution function with a well defined temperature T and drift velocity $\mathbf{u}$ [2]. This is the reason we can use the ideas of equilibrium thermodynamics to analyse the properties of strongly anisotropic systems, derive their thermodynamic functions and find quantities such as the energy density and the normal fluid density in the pulse.

One important question we address is, what is the range of stability of strongly anisotropic systems? The more anisotropic the phonon system is, the faster it travels: the velocity can closely approach the velocity of sound. We show that such systems are stable and do not break up into turbulence.

The pulse is predominantly composed of low energy phonons, l-phonons. The loss of energy from the phonon pulse is due to the creation of high energy phonons, h-phonons, which are subsequently lost from a short pulse because their group velocity is less than that of the pulse. Their energy $\varepsilon \geqslant 10$ K. As the pulse length is increased, the h-phonons stay longer in the pulse and come into equilibrium with the l-phonons. The density of h-phonons then is independent of the pulse length. This behaviour is seen experimentally [3]: the number of detected h-phonons increases with pulse length when all the created h-phonons are lost from the pulse without scattering, but the number saturates when the h-phonons are scattered so that they stay within the pulse. We have argued that the h-phonon density is much higher than in a cone, with the same total energy density and a Bose–Einstein distribution for an isotropic system, on dynamical grounds [4]. We shall see that this result can also be derived thermodynamically.

The evolution of such a phonon system is determined by its spatial size, energy and momentum densities and by the phonon dispersion curve. The dispersion curve initially rises faster than linearly [5, 6], which means that the phonons can interact by the three phonon process (3pp) [7]. Such interactions establish a phonon equilibrium, which is instantaneous on the scale of all the other times in the experiments. However, for phonons with momentum $p > p_c$ (h-phonons), the dispersion curve falls below a linear relation and so the fastest process is the four phonon process (4pp). This is about two orders of magnitude slower than 3pp, in strongly anisotropic systems.

Hence the phonon system can be divided into two subsystems: l-phonons and h-phonons. The h-phonons only weakly interact with the l-phonons. Furthermore, for sufficiently short pulses the h-phonons are lost from the main l-phonon pulse without any interaction. The creation and evolution of h-phonons in short pulses was investigated in [8, 9]. In contrast, for the long pulses, the h-phonons are in dynamic equilibrium with the subsystem of l-phonons. In practice the h-phonon spectrum has a maximum momentum p_f and all the phonons with $p < p_f$ are in quasi-equilibrium. This is determined by their creation rate, which strongly decreases with momentum, and the duration of the pulse, and scattering within the pulse.

The main aim of this paper is to calculate the thermodynamic functions of phonon systems, with any level of anisotropy, which are in equilibrium for phonon momenta up to p_f. An expression for the free energy of an anisotropic phonon system is obtained, and this enables us to calculate the thermodynamic functions of this system. These results show that strongly anisotropic phonon systems are essentially different from isotropic ones. The stability region for the phonon system is derived from general thermodynamic inequalities, and we show that strongly anisotropic phonon systems are stable over a wide temperature range.

The results presented here are not only interesting for describing phonon pulses in He II, but should be applicable to other anisotropic quasiparticle systems, in particular for phonon pulses in solids.

2. Distribution functions of anisotropic phonon systems

Consider an equilibrium anisotropic phonon system that includes phonons with momentum up to p_f. Unlike isotropic systems, the total momentum density $\mathbf{j}_0$ is not equal to zero. The distribution function for such an anisotropic phonon system, which makes the phonon–phonon collision integrals equal to zero, is the Bose–Einstein distribution with a drift velocity $\mathbf{u}$ that is parallel to $\mathbf{j}_0$:

$$n(\mathbf{p}) = \left[\exp\left(\frac{\varepsilon(\mathbf{p}) - \mathbf{p}\cdot\mathbf{u}}{k_B T}\right) - 1\right]^{-1} \tag{1}$$

where $\varepsilon(\mathbf{p})$ is the energy–momentum relation for phonons in superfluid helium. The pulse velocity can be calculated from $\mathbf{u}$ and T [10].

The energy–momentum relation can be written as

$$\varepsilon(\mathbf{p}) = cp(1 + \psi(p)), \tag{2}$$

where $c = 238$ m s^{-1} is the velocity of sound in helium, and the function $\psi(p)$, which describes the deviation from linearity, is small ($|\psi(p)| \ll 1$). Nevertheless it completely determines the type and strength of phonon interactions. According to [6], at low pressure, the function $\psi(p)$ is positive for phonons with momentum $0 < p < p_c$, where $cp_c/k_B = 10$ K at zero pressure. This is the momentum range of the l-phonons where the scattering is very fast. For small p the function $\psi(p) \sim p^2$. At $cp/k_B \approx 7$ K the function $\psi(p)$ reaches its maximum value ≈0.04. After that the function $\psi(p)$ decreases and becomes zero at $p = p_c$. For $p > p_c$, the momentum range for h-phonons, $\psi(p)$ is negative, so 3pp is prohibited by the conservation laws. In this case the fastest scattering is 4pp. An analytical approximation of the function $\psi(p)$ is given below (see equation (6)).

The distribution function can be rewritten using equations (1) and (2) as

$$n(p,\zeta) = \left[\exp\left(\frac{cp}{k_B T}(\chi + \psi(p) + \zeta(1-\chi))\right) - 1\right]^{-1}, \tag{3}$$

where $\zeta = 1 - \cos\theta$, θ is the angle between momentum $\mathbf{p}$ and the direction of $\mathbf{u}$, and $\chi = 1 - u/c$.

Until recently there only existed a theory of weakly anisotropic phonon systems, which assumed that $u \ll c$ (see, for example, [11]). That theory cannot describe strongly anisotropic phonon systems, in which $\chi \ll 1$, which are created in pulse experiments (see, for example, [1, 8]). The main aim of this paper is to give a theoretical description of phonon systems with any level of anisotropy, i.e. when the drift velocity takes all allowed values. For phonon system with $\chi \ll 1$ the distribution function (3) is highly anisotropic with a sharp maximum at $\zeta = 0$ and it has the typical width $\zeta \approx \chi + \psi_{\rm typ}$. Moreover, when $\chi \leqslant |\psi(p)|$ the distribution function (3) depends strongly on the function $\psi(p)$.

The anisotropy of a phonon system, with distribution function (3), can be described by a normalized angular distribution

$$W(\zeta) = \frac{\int \varepsilon n(p,\zeta) p^2\,\mathrm{d}p}{\int_0^2 \mathrm{d}\zeta \int \varepsilon n(p,\zeta) p^2\,\mathrm{d}p}. \tag{4}$$

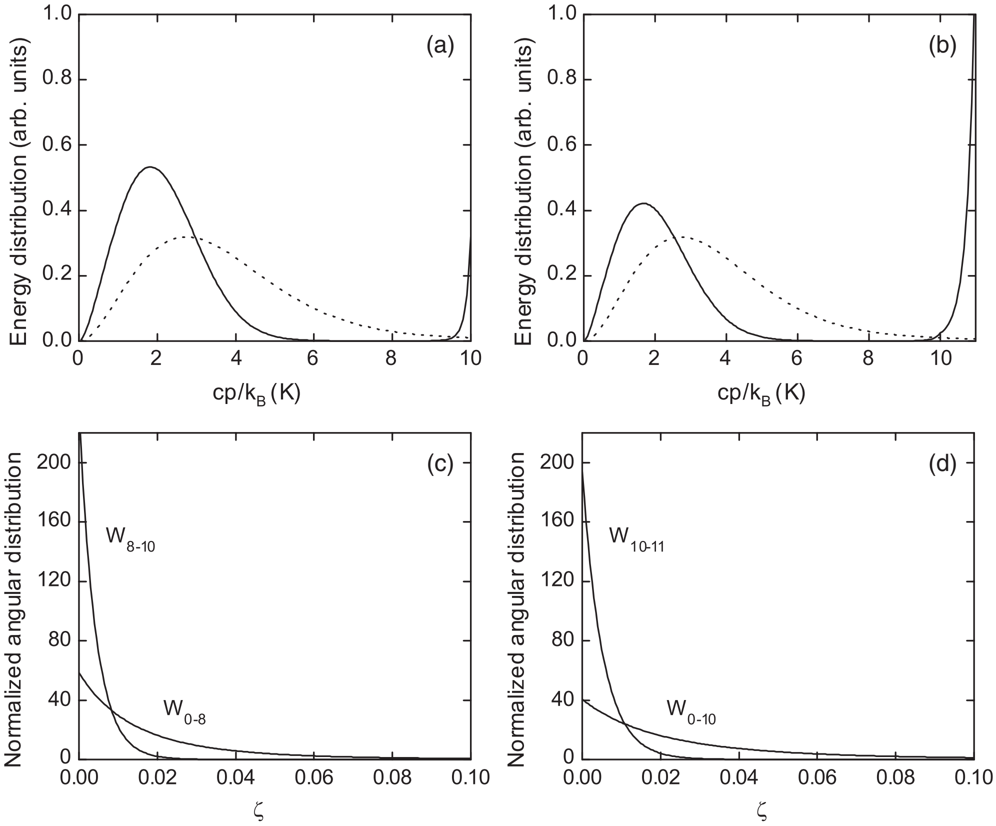

Figure 1. ((a) and (b)) The momentum dependence of the phonon energy distribution of strongly anisotropic phonon systems (solid lines), for $cp_f/k_B = 10$ K ($\chi = 0.02$ and $T = 0.041$ K), and $cp_f/k_B = 11$ K ($\chi = 0.042$ and $T = 0.054$ K), respectively. The dashed curves show the phonon energy distribution in a Bose cone ($\zeta_p = 0.023$ and $T_p = 1$ K). All distributions have the same energy and momentum densities. (c) and (d) show the normalized angular dependences of phonons calculated by equation (4). (c) shows the angular distribution $W_{0-8}(\zeta)$ of phonons, which form the first maximum in (a), where $0 < cp/k_B < 8$ K, and the angular distribution $W_{8-10}(\zeta)$ of phonons from the second maximum in (a), where $8 < cp/k_B < 10$ K, where for both $cp_f/k_B = 10$ K. (d) shows the angular distribution $W_{0-10}(\zeta)$ of phonons, which form the first maximum in (b), where $0 < cp/k_B < 10$ K, and the angular distribution $W_{10-11}(\zeta)$ of phonons which form the second maximum in (b), for $10 < cp/k_B < 11$ K, where for both $cp_f/k_B = 11$ K.

The momentum dependence of the energy distribution function is

$$\int_0^2 \varepsilon n(p,\zeta) p^2 \,\mathrm{d}\zeta = -\frac{k_B T p \varepsilon}{u}\left\{\ln\left[1-\exp\left(-\frac{\varepsilon(p)-pu}{k_B T}\right)\right] - \ln\left[1-\exp\left(-\frac{\varepsilon(p)+pu}{k_B T}\right)\right]\right\}. \tag{5}$$

The energy distribution function, equation (5), is shown in figures 1(a) and (b) for the l-phonon system with $cp_f/k_B = 10$ K, and for the (l + h)-phonon system with $cp_f/k_B = 11$ K, respectively.

The value of cp_f/k_B cannot be determined exactly but a number of arguments lead to the conclusion that it is up to 10 K for short pulses and around 11 K for long pulses. For short pulses, h-phonons are lost from the pulse in less than $\sim 5 \times t_p$ because their group velocity is less than that of the l-phonons ($\leqslant$189 and 238 m s^{-1} respectively). So for short pulses the h-phonons are lost from the back of the pulse before they come into equilibrium with the pulse.

Phonons with energy $\varepsilon/k_B \leqslant 8.9$ K [12] can interact, essentially instantaneously, by 3pp. There is some uncertainty in the value of this energy as it is sensitive to the exact form of $\psi(p)$ obtained from neutron scattering. Between $\varepsilon/k_B = 8.9$ and 10 K, 3pp are forbidden. A phonon in this energy range can decay into three or more phonons and similarly can be created from many phonons. The rates for such processes have not been calculated, but experiment [13] shows that the rate is less than the 3pp rate but higher than the 4pp rate. So for a short pulse, the h-phonons are not in equilibrium with the l-phonons because the h-phonons are lost from the pulse before they can interact with the l-phonons. However the l-phonons rapidly mutually interact up to $\varepsilon/k_B = 10$ K, so for this case we take $cp_f/k_B = 10$ K.

For a long pulse, there is time for the h-phonons to be in equilibrium with the l-phonons so p_f is higher than 10 K. Calculations of the creation rate of h-phonons show that the rate falls very quickly with increasing momentum; the rate for $cp/k_B = 11$ K is $\sim 10^{-2}$ of the rate for $cp/k_B = 10$ K, and the rate for $cp/k_B = 12$ K is $\sim 10^{-4}$ of the rate for $cp/k_B = 10$ K. As the rate rapidly decreases with momentum, we expect that cp_f/k_B is in the range 10–11 K. Measurements of the h-phonon pulse shapes indicate that for longer pulses, up to 350 ns, while most h-phonons have energy $cp/k_B \sim 10$ K there are phonons with $cp/k_B \sim 11$ K, [3]. In this paper we take $cp_f/k_B = 11$ K for long pulses, as a reasonable value for numerical calculations; however, the equations are correct for all values of p_f.

In figure 1 we use the following values: $\chi = 0.020$, $T = 0.041$ K at $cp_f/k_B = 10$ K for figures 1(a), (c), solid line; and $\chi = 0.042$, $T = 0.054$ K for $cp_f/k_B = 11$ K for figures 1(b), (d), solid line. These pairs of values of χ and T for $cp_f/k_B = 10$ K and $cp_f/k_B = 11$ K give the same values for the energy and momentum densities in the pulse. In the Bose-cone approximation, the numerical values $\chi = 0.020$, $T = 0.041$ K at $cp_f/k_B = 10$ K correspond to $\zeta_p = 0.023$ and $T_p = 1$ K, which are typical for experimental conditions [14], and are shown as dashed lines in figures 1(a) and (b). The values of the energy and momentum densities are the same as above, and correspond to a cone, with a solid angle of $\Omega_p = 2\pi\zeta_p$, cut from an isotropic Bose–Einstein distribution with temperature T_p. This Bose-cone approximation of the distribution function was used in previous analysis (see, for example, [15]).

For figure 1 and below we use the following parameterization for the function $\psi(p)$ at zero pressure, found by fitting to measured data [16]:

$$\begin{aligned} \psi(p < p_1) &= \gamma \frac{p^2}{p_c^2}\left(1 - \sigma \frac{p^2}{p_c^2}\right), \\ \psi(p > p_1) &= -\frac{p - p_c}{p}\left[\frac{c - c_h}{c} - \frac{\alpha}{2k_B}(p - p_c)\right], \end{aligned} \tag{6}$$

where the parameters are $\gamma = 0.181$, $\sigma = 1.13$, $cp_1/k_B = 8.26$ K, $c_h = 189$ m s^{-1} is the group velocity of the phonon with $cp_c/k_B = 10$ K and $\alpha = -19.8$ m s^{-1} K^{-1}.

Figures 1(a), (b) show the energy distribution function (5) as a solid line. It has two maxima in contrast to that for the Bose-cone distribution, shown as a dashed line, which has one maximum at $\varepsilon \approx 2.7T_p$. The second maximum is due to the function $\psi(p)$ which from $cp/k_B >\sim 7$ K decreases to zero; from (3) we see that this increases $n(p)$ when χ and ζ are small. The second maximum indicates that there is a large number of high energy phonons in a strongly anisotropic phonon system. The possibility of such an unusual distribution was first discussed in [4], where it was called a 'suprathermal distribution'.

Figures 1(c) and (d) show the normalized angular dependences, calculated using equation (4), for phonons that form the first maximum (figure 1(a), $0 < cp/k_B < 8$ K;

figure 1(b), $0 < cp/k_B < 10$ K), and the second maximum (figure 1(a), $8 < cp/k_B < 10$ K; figure 1(b), $10 < cp/k_B < 11$ K). These figures show that the high-energy phonons, which form the second maximum, have less angular width than those forming the first maximum. The narrower angular distribution of h-phonons, compared to l-phonons, was observed in experiments [17].

3. Thermodynamic functions of anisotropic phonon system

The thermodynamic properties of a phonon system can be determined from the free energy density; see [11]:

$$F = k_B T \int \ln\left[1 - \exp\left(-\frac{\varepsilon(\mathbf{p}) - \mathbf{p}\cdot\mathbf{u}}{k_B T}\right)\right] \frac{d^3 p}{(2\pi\hbar)^3}. \tag{7}$$

The integration over angle in equation (7) can be done if the logarithm, in the integral expression, is expanded in a series and then each term of the series integrated. As the result we obtain

$$\begin{aligned} F = \frac{k_B T c}{8\pi^2\hbar^3 u} \int_0^{p_f} \Bigg\{ &\left(1 - \frac{u}{c} + (p\psi(p))'\right) \ln\left[1 - \exp\left(-\frac{\varepsilon(p) - pu}{k_B T}\right)\right] \\ &- \left(1 + \frac{u}{c} + (p\psi(p))'\right) \ln\left[1 - \exp\left(-\frac{\varepsilon(p) + pu}{k_B T}\right)\right]\Bigg\} p^2 \, dp \\ &- \frac{k_B^2 T^2 p_f^2}{8\pi^2\hbar^3 u} \sum_{n=1}^{n=+\infty} \frac{1}{n^2} \left\{\exp\left(-\frac{\varepsilon_f - p_f u}{k_B T} n\right) - \exp\left(-\frac{\varepsilon_f + p_f u}{k_B T} n\right)\right\}. \end{aligned} \tag{8}$$

For weakly anisotropic systems, when $u \ll c$, the upper limit p_f in expression (7) can be changed to infinity and the function $\psi(p)$ can be taken as zero. The integration then gives the result in [11].

The thermodynamic functions of a phonon gas can be obtained by differentiating the free energy density F. So, the entropy density S, heat capacity density C, the momentum density $\mathbf{j}_0$, and the density of the normal component ρ_n are given respectively by

$$S = -\frac{\partial F}{\partial T}, \qquad C = -T\frac{\partial^2 F}{\partial T^2}, \qquad \mathbf{j}_0 = -\frac{\partial F}{\partial \mathbf{u}}, \qquad \rho_n = \frac{j_0}{u}. \tag{9}$$

The signal amplitude on a bolometer due to a phonon pulse is proportional to the energy density E of the phonon system. The energy density can be written in terms of the thermodynamic functions by the following relation:

$$E = F + TS + \mathbf{j}_0 \cdot \mathbf{u}. \tag{10}$$

However, it is easier to use

$$E = \int \varepsilon n \frac{d^3 p}{(2\pi\hbar)^3}. \tag{11}$$

We now expand the distribution function n in equation (11) in a series in powers of $\exp(-(\varepsilon(\mathbf{p}) - \mathbf{pu})/k_B T)$, and then integrate each term of the series over the ζ. Finally, summing the resulting series we find

$$E = -\frac{k_B T}{4\pi^2\hbar^3 u} \int_0^{p_f} dp \, p\varepsilon \left\{\ln\left[1 - \exp\left(-\frac{\varepsilon(p) - pu}{k_B T}\right)\right] - \ln\left[1 - \exp\left(-\frac{\varepsilon(p) + pu}{k_B T}\right)\right]\right\}. \tag{12}$$

For weakly anisotropic phonon systems, when $u \ll c$, in the rhs of equation (12), $\psi(p)$ can be considered equal to zero, and so the upper limit p_f can be taken as infinity. Finally, we obtain

$$E = \frac{\pi^2 (k_B T)^4}{30\hbar^3 c^3} \frac{1 + u^2/3c^2}{(1 - u^2/c^2)^3}, \qquad \psi(p) = 0. \tag{13}$$

For strongly anisotropic phonon systems, when $\chi \ll 1$, one can omit the second term in the braces in the integral expression equation (12). Then the integral for the energy can be considered as a sum of two integrals, corresponding to phonons in the first maximum, $E_{\rm I}$, and in the second maximum, $E_{\rm II}$. As the phonons in the region of the first maximum in equation (12) have low energies, one can use the following quadratic approximation $\psi(p) = \gamma p^2/p_c^2$, and take the upper limit of integration as infinity. As a result we obtain

$$E_{\rm I} = \frac{(k_B T)^4}{4\pi^2 (\hbar c \chi)^3 (1 - \chi)} I\left(\frac{\gamma k_B^2 T^2}{p_c^2 c^2 \chi^3}\right), \tag{14}$$

where we use notation

$$I(x) = -\int_0^{+\infty} dy\, y^2 \ln[1 - \exp(-y - xy^3)]. \tag{15}$$

For the limiting cases $x \gg 1$ and $x \ll 1$ one can get asymptotic formulae for the integral (15):

$$I(x) = \frac{\pi^4}{45}\left\{1 - \frac{40\pi^2}{7} x\right\}, \qquad x \ll 1; \tag{16}$$

$$I(x) = \frac{\pi^2}{18x}\left\{1 - \frac{2\Gamma(\frac{1}{3})\zeta(\frac{4}{3})}{\pi^2 \sqrt[3]{x}}\right\}, \qquad x \gg 1; \tag{17}$$

where $\Gamma(1/3) \approx 2.679$ and $\zeta(4/3) \approx 3.601$.

The first term of the asymptotic expansion equation (16) gives the dependence $E_{\rm I} \sim T^4$, valid for very low temperatures $T \leqslant 0.006$ K when $\chi = 0.02$, i.e.,

$$E_{\rm I} = \frac{\pi^2 (k_B T)^4}{180 (\hbar c \chi)^3 (1 - \chi)}. \tag{18}$$

For the asymptotic equation (17) we obtain a dependence which is close to $E_{\rm I} \sim T^2$, which is valid for higher temperatures, $T \geqslant 0.18$ K when $\chi = 0.01$, i.e.,

$$E_{\rm I} = \frac{p_c^2 (k_B T)^2}{72\hbar^3 c \gamma (1 - \chi)}\left\{1 - \frac{2\Gamma(\frac{1}{3})\zeta(\frac{4}{3})(c p_c)^{2/3} \chi}{\pi^2 \gamma^{1/3} (k_B T)^{2/3}}\right\}. \tag{19}$$

For $\chi = 0.02$ the asymptotic equation (19) is formally valid if $T \geqslant 0.5$ K, but for such values of χ and T, the two maxima of the phonon energy distribution function overlap, so the approximation which was made when we obtained equation (14) is poor. Nevertheless, the asymptotic expression (19) is important, because it shows the limiting law for $E_{\rm I}$, which is valid for relatively large T and small χ.

To calculate $E_{\rm II}$, the contribution of the phonons in the second maximum to the energy density, we note that the integral expression in equation (12) has a sharp maximum at $p = p_f$ caused by the exponential function. This enables us to make several approximations: the slowly changing function $p\varepsilon$ can be substituted by cp_f^2, the exponent in equation (12), $(\varepsilon(p) - pu)$, can be expressed as a linear function of p at $p = p_f$, and the lower limit of integration can be taken as $-\infty$. These allows us to get the following result:

$$E_{\rm II} \approx \frac{p_f^2 (k_B T)^2}{4\pi^2 \hbar^3 c (1 - \chi)} \frac{1}{[-(p\psi(p))'|_{p=p_f} - \chi]} \exp\left\{-\frac{c p_f}{k_B T}[\chi + \psi(p_f)]\right\}. \tag{20}$$

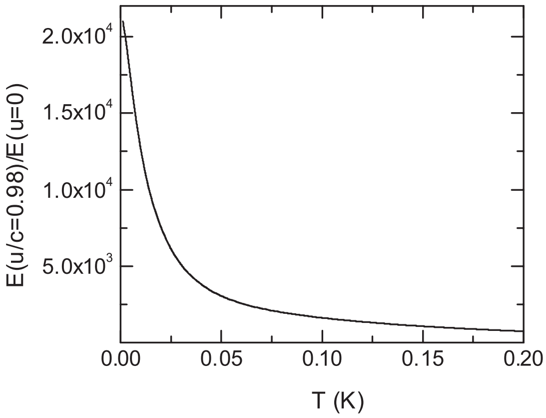

Figure 2. The ratio of the energy in the strongly anisotropic phonon system with $\chi = 0.02$ to that in the isotropic phonon system with $\chi = 1$, as a function of temperature.

With typical experimental values, the contribution of $E_{\rm II}$ to the total energy density is considerably less than that of $E_{\rm I}$.

For the case of the Bose-cone approximation, the energy density of the h-phonons, noting that $cp_c/k_{\rm B}T_{\rm p} \gg 1$, can be written as follows:

$$E_{\rm II}^{(\rm B.c.)} \approx \frac{2\pi\zeta_{\rm p}(k_{\rm B}T_{\rm p})^4}{(2\pi\hbar c)^3}\left\{ {\rm e}^{-\varepsilon_c/k_{\rm B}T_{\rm p}}\left[\left(\frac{\varepsilon_c}{k_{\rm B}T_{\rm p}}\right)^3 + 3\left(\frac{\varepsilon_c}{k_{\rm B}T_{\rm p}}\right)^2\right] - {\rm e}^{-\varepsilon_f/k_{\rm B}T_{\rm p}}\left[\left(\frac{\varepsilon_f}{k_{\rm B}T_{\rm p}}\right)^3 + 3\left(\frac{\varepsilon_f}{k_{\rm B}T_{\rm p}}\right)^2\right]\right\}. \qquad (21)$$

To compare the energy densities of h-phonons in anisotropic and in Bose-cone systems we evaluate expressions (20) and (21), at the same value of p_f, and we take pairs of χ, T and $\zeta_{\rm p}, T_{\rm p}$, which correspond to the same energy and momentum densities. So, at $cp_f/k_{\rm B}T = 11$ K, we take $\chi = 0.042$ and $T = 0.054$ K, which correspond to the typical experimental values, $\zeta_{\rm p} = 0.023$ and $T_{\rm p} = 1$ K. The ratio of energy densities, determined by (20) and (21), at these values, is equal to

$$\frac{E_{\rm II}}{E_{\rm II}^{(\rm B.c.)}} \approx 33. \qquad (22)$$

The result (22) is close to that obtained in [18] which started from the relation between the rates of creation and decay of h-phonons in the l-phonon pulse in the Bose-cone approximation.

Figure 2 shows the temperature dependence of the ratio of the total energy in the strongly anisotropic phonon system with $\chi = 0.02$, and the isotropic phonon system with $\chi = 1$. Figure 2 shows that the total energy density of the strongly anisotropic phonon system is always greater than the total energy density of the isotropic phonon system at the same temperature. The ratio of the total energy densities $E_{\rm I}$ (equation (14)) to E (equation (13) with $u = 0$) increases with decreasing temperature and reaches its maximum value $1/(6\chi^3(1-\chi))$ at $T = 0$.

4. The stability of anisotropic phonon systems

According to [19], superfluid motion in He II is thermodynamically stable if the following equality is satisfied:

$$\left(\frac{\partial P}{\partial \rho}\right)_{T,u}\left[\rho-\left(\frac{\partial j_0}{\partial u}\right)_{T,\rho}\right]-\rho\left[\left(\frac{\partial j_0}{\partial \rho}\right)_{T,u}-u\right]^2>0, \tag{23}$$

where P is the pressure of helium.

The inequality (23) can be considered as a generalization of the Landau criterion for finite temperatures. As it will be shown later, up to temperatures 2 K, the values of the drift velocity u, on the stability curve, which divides the stable and unstable regions, is very close to c; i.e. such phonon systems must be considered as strongly anisotropic as $\chi \ll 1$. After differentiating equation (8) for the free energy twice with respect to u, and omitting small terms, we get the approximate relation:

$$\frac{1}{\rho}\left(\frac{\partial j_0}{\partial u}\right)_{T,\rho}\approx\frac{1}{4\pi^2\hbar^3\rho c}\int_0^{p_f}\frac{p^3\,\mathrm{d}p}{\exp(\frac{cp}{k_BT}[\chi+\psi(p)])-1}. \tag{24}$$

The momentum density j_0 depends on the helium density ρ, because the phonon spectrum depends on density:

$$\frac{\partial\varepsilon}{\partial\rho}=\frac{w}{\rho}\varepsilon+cp\frac{\partial\psi(p)}{\partial\rho}\approx\frac{w}{\rho}\varepsilon, \tag{25}$$

where $w=(\rho/c)(\partial c/\partial\rho)=2.84$ is the Gruneisen constant. The experimental data show that the contribution of the second term can be neglected.

Then we differentiate expression (8) for the free energy with respect to u and with respect to ρ, and obtain the approximate relation for strongly anisotropic systems:

$$\frac{1}{c}\left(\frac{\partial j_0}{\partial \rho}\right)_{T,u}\approx -w\frac{1}{\rho}\left(\frac{\partial j_0}{\partial u}\right)_{T,\rho}. \tag{26}$$

Substituting the equalities (24) and (26) into the stability criterion (23), and noting $\chi \ll 1$, we find

$$\frac{1}{4\pi^2\hbar^3\rho c}\int_0^{p_f}\frac{p^3\,\mathrm{d}p}{\exp(\frac{cp}{k_BT}(\chi+\psi(p)))-1}<\frac{\chi}{w+1/2}. \tag{27}$$

The inequality (27) determines the region of thermodynamic stability of phonon systems; it is stable when $u<u_{st}(T)$. Figure 3 shows the dependence of the critical velocity, $u=u_{st}(T)$, on temperature when $cp_f/k_B=10$ K and $cp_f/k_B=11$ K, calculated using equation (27). At $T=0$ K, the value of u_{st} coincides with the Landau critical velocity for a phonon system; it is the condition that the distribution function must be positive for all possible values of phonon momenta:

$$\chi>\max_{p\in[0,p_f]}[-\psi(p)]=-\psi(p_f),\qquad \text{if } cp_f/k_B\geqslant 10\text{ K}. \tag{28}$$

It can be seen from figure 3 that the critical velocity decreases monotonically with increasing temperature, but remains close to the velocity of sound, c. So, strongly anisotropic phonon systems are stable thermodynamically in the wide temperature range up to ~2 K. Figure 4 shows the temperature dependences of ρ_n/ρ for the most extreme anisotropic systems, i.e. on the stability line $u=u_{st}(T)$ for $cp_f/k_B=10$ K, and for the isotropic case $u=0$.

In figure 4 one can see that for strongly anisotropic phonon systems the normal density ρ_n can strongly exceed the normal density of the isotropic case, while remaining small compared to the total density of helium ρ.

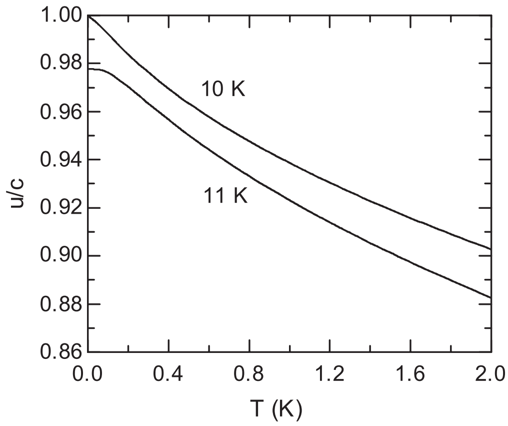

Figure 3. The critical velocity, $u = u_{\mathrm{st}}(T)$, for $cp_f/k_B = 10$ K and $cp_f/k_B = 11$ K, as a function of temperature.

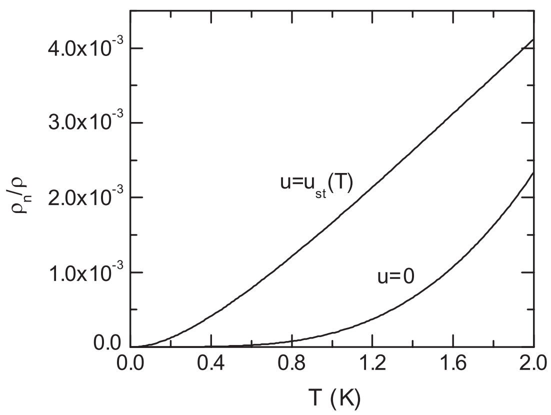

Figure 4. The normal fluid density, normalized to the total density, ρ_n/ρ, for the extremely anisotropic phonon system on the stability line $u = u_{\mathrm{st}}(T)$ with $cp_f/k_B = 10$ K, and for the isotropic system $u = 0$, as a function of temperature.

Figure 5 shows the ratio of the total energy density of a strongly anisotropic phonon system on the stability line $u = u_{\mathrm{st}}(T)$, for $cp_f/k_B = 10$ K, to the energy density of the isotropic phonon system as a function of temperature. At 2 K this ratio is 5.2. We see that for sufficiently low temperatures, the energy density of a strongly anisotropic phonon system can be many orders greater than the density of the isotropic system at the same temperature.

5. Conclusion

In this paper we have investigated anisotropic phonon systems, in thermodynamic equilibrium, that are characterized by the Bose–Einstein distribution function which includes the temperature T and the drift velocity $\mathbf{u}$ (see equation (1)). For strongly anisotropic phonon systems, the phonon energy distribution function can have two maxima, in contrast to one for the isotropic case (see figures 1(a), (b)). The second maximum results from the fact that at $p = p_f$ the function $\psi(p)$, which determines the deviation of the energy–momentum relation

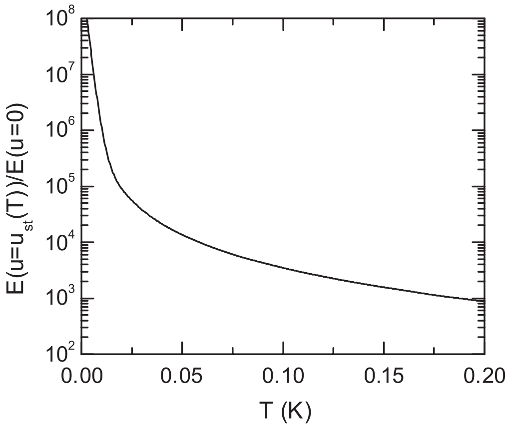

Figure 5. The ratio of the total energy density of an extremely anisotropic phonon system on the stability line $u = u_{st}(T)$, with $cp_f/k_B = 10$ K, to the energy density of the isotropic phonon system $u = 0$, as a function of temperature.

from a linear dependence, reaches its minimum value. Phonons that form the second maximum have momentum close to the maximum momentum p_f, and their angular width is much less than the angular width of the phonons which form the first maximum (see figures 1(c), (d)). This is in agreement with experimental results [14].

An expression for the free energy of the anisotropic phonon system is found (8) which allows all the thermodynamic functions of the system to be calculated (9). In the limiting case of a weakly anisotropic phonon system, when $u \ll c$, the equations give the well-known relations from [11]. For strongly anisotropic systems we obtain new results. In particular, it was shown that the energy density of strongly anisotropic phonon systems at low T increases as T^4 (18), and for higher temperatures it increases more slowly, as T^2 (19), but only if the contribution of the second maximum (20) is less than that of the first maximum (14). For typical experimental values, the contribution of the second maximum is always less than the contribution of the first maximum. But at the same time, the energy of these h-phonons can greatly exceed the energy of the h-phonons for the Bose-cone distribution. The ratio of these energies may reach the suprathermal value of several tens (see equation (22)). So, phonon–phonon interactions result in a large number of phonons with high momentum when the system is strongly anisotropic, i.e. an increase in the density of h-phonons with anisotropy. In a previous paper this result was obtained by an analysis of the creation and decay rates of h-phonons with the approximate Bose-cone distribution function [4]. From the limiting equation (18), we see that strongly anisotropic phonon systems with $\chi \ll 1$ and low temperatures have a higher total energy density than isotropic phonon systems at the same temperature (see figure 2).

Realizable anisotropic phonon systems must be thermodynamically stable. The general thermodynamic inequality [19] that can be applied to superfluid helium determines the line of stability $u = u_{st}(T)$. We found the stability line for quasi-equilibrium systems of l- and (l + h)-phonons (see figure 3). At $T = 0$ the maximum drift velocity coincides with the Landau velocity (see equation (28)); hence, the curves $u = u_{st}(T)$ can be considered as a generalization of the critical Landau velocity of phonon systems at finite temperatures. For temperatures up to $T = 2$ K the critical velocity remains close to c, so strongly anisotropic phonon systems are stable up to high temperatures.

We have shown (see figure 4) that, for strongly anisotropic phonon systems, the normal density ρ_n can be much higher than the normal density of isotropic phonons at the same temperature, but it remains small in comparison with the total density of helium, ρ. The total energy density (figure 5) of extremely anisotropic phonon systems, at sufficiently low temperatures, can be many orders greater than the energy density of the isotropic system at the same temperature.

Finally, we note that there is an essential difference between anisotropic phonon systems and anisotropic systems of classical particles. In the latter it is possible to transform the anisotropy away, by making a Galilean transformation to a moving frame where the total momentum of the system is zero. This is impossible for the anisotropic phonon system because the energy–momentum dependence, $\varepsilon(p)$, is only true in the frame in which the superfluid is at rest: in a classical system the energy–momentum relation is the same in all inertial frames. So, for phonon systems in superfluid helium, the anisotropy is absolute.

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References

[1] Wyatt A F G, Lockerbie N A and Sherlock R A 1989 *J. Phys.: Condens. Matter* **1** 3507
[2] Adamenko I N, Kitsenko Yu A, Nemchenko K E, Slipko V A and Wyatt A F G 2005 *Phys. Rev.* B **72** 054507
[3] Vovk R V, Williams C D H and Wyatt A F G 2004 *Phys. Rev.* B **69** 144524
[4] Wyatt A F G, Tucker M A H, Adamenko I N, Nemchenko K E and Zhukov A V 2000 *Phys. Rev.* B **62** 3029
[5] Maris H J and Massey W E 1970 *Phys. Rev. Lett.* **25** 220
[6] Stirling W G 1983 *75th Jubilee Conf. on Liquid Helium-4* ed J G M Armitage (Singapore: World Scientific) p 109
[7] Havlin S and Luban M 1972 *Phys. Lett.* A **42** 133
[8] Tucker M A H and Wyatt A F G 1994 *Physica* B **194** 549
[9] Adamenko I N, Nemchenko K E, Zhukov A V, Tucker M A H and Wyatt A F G 1999 *Phys. Rev. Lett.* **82** 1482
[10] Adamenko I N, Nemchenko K E, Slipko V A and Wyatt A F G 2005 *J. Phys.: Condens. Matter* **17** 2859
[11] Khalatnikov I M 1965 *An Introduction to the Theory of Superfluidity* (New York: Benjamin)
[12] Adamenko I N, Kitsenko Yu A, Nemchenko K E, Slipko V A and Wyatt A F G 2005 *Fiz. Nizk. Temp.* **31** 607
Adamenko I N, Kitsenko Yu A, Nemchenko K E, Slipko V A and Wyatt A F G 2005 *Low Temp. Phys.* **31** 459 (Engl. Transl.)
[13] Mezei F, Lartigue C and Farago B 1991 *Excitations in Two-Dimensional and Three-Dimensional Quantum Fluids* ed A F G Wyatt and H J Lauter (New York: Plenum) p 119
[14] Tucker M A H and Wyatt A F G 1994 *J. Phys.: Condens. Matter* **6** 2813
[15] Adamenko I N, Nemchenko K E and Wyatt A F G 2002 *J. Low Temp. Phys.* **126** 609
[16] Adamenko I N, Nemchenko K E and Wyatt A F G 2001 *J. Low Temp. Phys.* **125** 1
[17] Tucker M A H and Wyatt A F G 1994 *Physica* B **194** 551
[18] Adamenko I N, Nemchenko K E and Wyatt A F G 2003 *Fiz. Nizk. Temp.* **29** 16
Adamenko I N, Nemchenko K E and Wyatt A F G 2003 *Low Temp. Phys.* **29** 11 (Engl. Transl.)
[19] Andreev A F and Melnikovsky L A 2003 *Pis. Zh. Eksp. Teor. Fiz.* **78** 1063
Andreev A F and Melnikovsky L A 2003 *JETP Lett.* **78** 574 (Engl. Transl.)